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Interdependent Security Game Design over Constrained Linear Influence Networks

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Abstract

In today's highly interconnected networks, security of the entities are often interdependent. This means security decisions of the agents are not only influenced by their own costs and constraints, but also are affected by their neighbors' decisions. Game theory provides a rich set of tools to analyze such influence networks. In the game model, players try to maximize their utilities through security investments considering the network structure, costs and constraints, which have been set by the network owner. However, decisions of selfish entities to maximize their utilities do not always lead to a socially optimum solution. Therefore, motivating players to reach the social optimum is of high value from the network owner's point of view. The network owner wants to maximize the overall network security by designing the game's parameters. As far as we know, there is no notable work in the context of linear influence networks to introduce appropriate game design for this purpose. This paper presents design methods that make use of the adjustments of players' costs, interdependencies, and constraints to align players' incentives with a network-wide global objective. We present a comprehensive investigation of existence and uniqueness conditions of Nash Equilibrium in such environments. Furthermore, numerical results of applying the proposed mechanisms in a sample real-world example are illustrated.

1 Introduction

Today's tightly interconnected systems lead to a high degree of interdependency in system security. This means that security of each part depends on its own security investment as well as the investments made by its neighbors along with the structure of their connections. The impact of neighboring agents' actions and similar networking effects on individual agent security can be described using the framework of linear influence networks as a first order approximation [1]. In such interdependent networks the amount of security investments of players are usually constrained by a variety of factors such as budgeting, depending on the network owner attitude. These constraints as well as the network structure play a vital role in how participants eventually decide. In such environments, the agents' rational and selfish decisions may produce non-optimal results from the

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system point of view [2, 3]. Therefore, the design of such networks for achieving better global outcomes is an important and interesting research question. This topic, which also encapsulates different features of "grid", "utility", and "cloud" computing, has driven lots of interest in the interdependent security world [4].

As a real world example we can refer to different sectors of a company running their applications on software and hardware resources provided by the company's service provider which might be partly or completely shared between different sectors. In different sectors, use of shared software/hardware components can make the entire system susceptible to security incident by a single vulnerability. In other words, when different sectors use the same software infrastructure like a specific operating system (OS), security investments of one sector (i.e. using anti-malware or patching the vulnerabilities of its own application running to make it more secure in case of an attack) may have a significant impact on the security of other sectors. This type of impact is known as the externalities in the literature to refer to the effects of network nodes' investments on each other. In positive externalities, an increase in security level of a node as a result of its investment, positively affects the security of other nodes by providing extra protection via the deployed security of the node. On the other hand, in case of negative externalities, the increased security level of a node inversely leads to increase the risks of others by encouraging the attacker to switch its target to weaker nodes.

The field of game theory provides a solid mathematical framework for interactions and decision modeling of agents on linear influence networks [5]. A primary research topic is the investigation of Nash Equilibrium (NE) outcomes, i.e. equilibrium points at which all players play their best possible actions known as their best responses and no player has an incentive to unilaterally deviate. In recent years there has been a growing interest in investigation of NE in linear interdependent networks, which are reviewed in Section 3. However, under many conditions, NE points are not globally optimum. Therefore, the problem of game design to achieve global objectives has been attracted interests during the last few years [6, 7]. For instance, in our case, investments of selfish players does not guarantee to maximize the total utility of the network. So, the mechanism to reach the global optimum is an interesting question.

In majority of the cases in classical game theory, games are designed by introducing a cost to the players' utilities to reach the global objective. In other words, the game designer knows or can compute the social optimum and aims to send players a price signal which makes them play according to the designer's desired strategy. This paper presents a similar design method as a starting point. As a contribution, however, we show that the game can also be designed by modifying other network parameters. This modification is implemented through the connection strength or constraints adjustments while keeping the original global objective, which poses an interesting research challenge. For example, in our case study, the designer by modifying the weights of the links in the interdependency graph or putting some budget constraints on the players can encourage the players to reach the social optimum. Thus, we propose a new game-theoretic framework for designing different network-related parameters to align the players' incentives to the system owner's security objectives in constrained linear influence networks.

When designing the game and mapping the equilibrium point to the socially defined optimum, it is very useful to have a unique NE solution for the specified game. There are some basic NE existence and uniqueness results on the linear influence networks [1, 8, 9]. However, these methods are not complete under the considered constrained cases in this paper. Therefore, a strong result on the existence of a unique pure NE is established in linear influence network games under some general constraints. Different sufficient conditions for ensuring uniqueness are also presented.

To show the applicability of the proposed design methods on enhancing security investment decisions in interdependent networked systems, we perform an analysis on a sample real world example. For this purpose, the considered case is modeled as a game and three different proposed mechanisms are applied to the game. As the result, we numerically analyze the movement of the game's Nash Equilibrium to the social optimum for each mechanism and compare the achieved gains together regarding their corresponding ease, cost and feasibility.

The **main contributions** of this paper include the following:

- A game design framework is developed to achieve the social objective on constrained linear influence networks by three different methods:
  (1) Modifying players' costs through price signals encourages them to approach the social optimum at the NE solution.
  (2) The design objective is achieved by direct modification of the linear influence graph. In other words, players can reach the social optimum by modifying the interdependencies.
  (3) The system constraint set is modified to

move the NE to the global optimum solution. In other words, players can reach the social optimum by modifying the constraints on the players' actions.
- We also find sufficient conditions for existence and uniqueness of NE in the games with general constraints on linear influence networks.

The rest of the paper is organized as follows: Section 2 presents a motivating example to clarify more the problem we have solved. Section 3 provides an overview of the related works. Section 4 presents the general structure of the game. The game design using costs modifications is presented in Section 5. In Section 6, the new game design using the weight modification as well as an illustrative numerical example are presented. Another new method of designing games using constraints modification is investigated in Section 7. In Section 8 the numerical results of applying design methods in a case study are presented. The conclusions are finally drawn in Section 9. Some existence and uniqueness results for general constraints and linear influence networks are also presented in Appendix A.

2 Motivating Example

As a motivating example in real-world, let us consider a company with different sectors utilizing the services and infrastructures provided by the company's private service provider. In these networks users take advantage of software and hardware resources the service provider offers. For instance, service provider usually share some subsets of hardware infrastructures like HDD and software components like virtualization operating system (OS), OS of hosts, database applications, and web server apps. Therefore, some levels of interdependency take place between users regarding their shared resources. In other words, by exploiting a vulnerability in a system, not only the system's own private information will be lost or affected by the virus or attacker, the shared resources of other interconnected nodes can also be affected. In these networks users invest on their security considering their limited budget and the effects of other users' security investments. In fact, this investment not only improves the node's security, but also promotes the security of other connected users proportional to their common resources.

On the other hand, the company can define and modify the level of players' interdependencies through the modification of its service provider settings. For example, the interdependencies of players can be modified in both directions when the service provider makes changes to the amount of shared resources. Moreover, the service provider can change the level of interdependency between two players in one direction by increasing or decreasing the recovery considerations (i.e. making back up files from a portion of information) for any one of them. In other words, in case a player is attacked and other player is affected regarding the existence and the amount of shared resources, if the service provider considers some recovery mechanisms for the second player, the interdependency of the second player is reduced as a result of recovery of its resources. Therefore, the interdependency of node j to node i can be defined as follows:

$$\mathbf{W}_{ij} = \frac{|\, \mathbf{R}_i \cap \mathbf{R}_j \,|}{|\, \mathbf{R}_i \cup \mathbf{R}_j \,|} - \mathit{RecoveryGuarantee}(\mathbf{R}_j) \quad (1)$$

In this formula, $\mathbf{R}_i$ and $\mathbf{R}_j$ represent resources used by players i and j respectively. The bidirectional interdependency of two players based on their shared resources can be computed as $\frac{|\mathbf{R}_i \cap \mathbf{R}_j|}{|\mathbf{R}_i \cup \mathbf{R}_j|}$, and the portion of resources of specific player j with recovery guarantee can be represented as $\mathit{RecoveryGuarantee}(\mathbf{R}_j)$.

Figure 1 presents sectors of a company which are making use of the company's private service provider. The usage of shared memory (the resource in our case) forms their interdependencies. In this example network, some levels of interdependency are defined between nodes regarding the amount of their shared memory usage, and as a result of the absence of recovery considerations in their initial configuration, all connections are defined bidirectional. For example, the amount 0.5 represented on the edges e_{41} and e_{14} shows that in the initial configuration, the service provider has shared half of the memory of nodes 1 and 4 without any recovery guarantee.

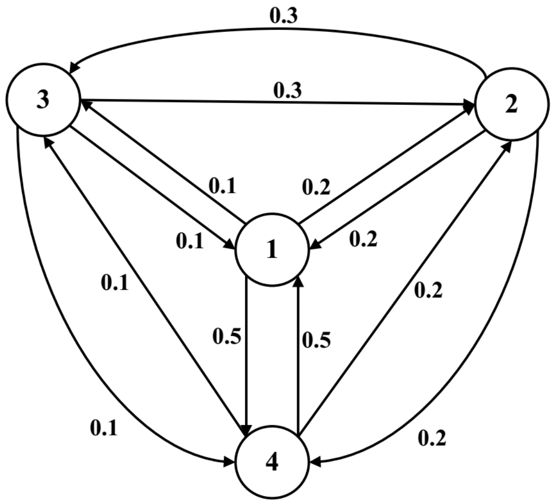

Figure 1. A company with four sectors which are interdependent with respect to their shared hardware/software resources.

In such networks, not only the players try to improve their security regarding their interdependencies and limited budget, but the total security of the network is also of highest importance for the company. Unfortunately, the players' selfish security investments considering their limited budget and interdependencies do not usually guarantee the company network to reach the highest total security in the equilibrium point of the game. Therefore, this motivating example raises a research challenge that how the company can improve the total security of the whole network by making sectors play accordingly.

- The service provider can charge the sectors with different amounts to be encouraged to reach the social optimum with the highest total security for the network. In this paper we apply this typical game design method to linear influence networks in Section 5.
- The service provider can also modify the level of interdependencies by eliminating the amount of common resources or applying some recovery mechanisms on the portion of shared resources of some sectors. In Section 6 we also present a new method in which by influence modification that might have lower side-effects on user satisfaction than charging them with money, encourage players to approach the social optimum.
- The company can also lead the sectors to reach the social optimum by applying some budgeting constraints on them. Therefore, the effect of applying security budgeting constraints on reaching the social optimum is studied in Section 7.

In the following, after proposing and investigating different mechanisms, in Section 8 we apply them to solve the problem raised here for the company whose network is presented in Figure 1.

3 Related Work

Applications of game theory to networked systems have attracted special attention in the literature [10, 11]. There are multiple models which consider several strategic decision-makers that are connected through the edges of a relational graph. Several studies have come to be studied that by considering information security as a product, utilize new economic approaches to analyze defensive security methods. [12–14] Applying new economic approaches to the information security problems led to study of Thief and Police two player strategic games between an attacker and a defender [15–18]. In modern networks, decisions of nodes (players) along with the topology and structure of the network affect the decisions of other interdependent nodes [19]. Hence, simple two-player information security games are replaced with the new class of games known as interdependent security games in which there are several selfish but non-malicious players compete to increase their utility by changing their security investments. In such games, the goal of each player is to maximize its utility which depends on its relevant costs and the investments made by other players [4].

Many studies in this field have taken symmetry into account for the participants, their interdependencies or utility functions. Galeotti *et al.* [20] proposed an interdependent investment game model in the economics literature which assumes nodes with incomplete information about the structure and topology of the network. The nodes then compute their utilities based on their degree in the network graph. A quasi-aggregative game was presented in [21], where the utility of each player depends on a concave function of player's own strategy along with some aggregation functions of its neighbors' strategies, from which the cost of the player's strategy is reduced. However, this model does not consider the effect of different network interdependencies through the edges weights. Bramoulle *et al.* [22] proposed a linear game-theoretic interdependent model with complete information. In their model, the player's utility, similar to the Galeotti's model, is calculated using a linear aggregative function of the investments made by its neighbors. However, in the aggregative function of each player, the neighbors' investments were affected by the edge weights of the network graph. They also investigated uniqueness and stability of NE using the eigenvalue of the graph (representing the network's structure). Based on these results, Perciado *et al.* [23] presented some analysis based on the eigenvalue of differently structured graphs.

Yolken and Miura-Ko [1, 8, 9] developed an interdependent security game based on Bramoulle's model. They not only proposed an aggregative linear function based on players' investments, but in contrast to the previous models, they also studied asymmetric and non-identical components, interdependencies, and utility functions. Utilizing the concept of linear influence networks, they considered the effect of network externalities within their game model. To investigate the existence and uniqueness of NE, they took advantage of the reduction to a Linear Complementarity Problem (LCP). Finally, they provided some convergence algorithms to the NE. By the same token, Ballester *et al.* [24] developed a similar method and modeled the game based on Linear Influence Networks whose NE is expressible through an LCP reduction. They used Katz centrality metric concept [25] to investigate the existence and uniqueness of NE in such networks. There are other different researches in which existence and uniqueness conditions of Nash equilibrium are studied [9, 14, 26–28]. However, none of the mentioned works considered constraints that limit players' actions in respective games analyzed. To the best of our knowledge, previous works have only been limited to existence and uniqueness investigation of NE in interdependent security networks without any or with very simple constraints. This paper proposes and studies new existence and uniqueness conditions taking into account the effect of general constraints on linear influence networks.

The closer the Nash equilibrium point to the globally optimal strategy profile, which comes from the independent decision-making of the selfish players, the more effective the equilibrium point would be. The social optimum in interdependent security games is generally defined as the minimum sum of the players' costs or the maximum sum of their utilities, known as social welfare. The review of the quality of Nash equilibrium points in interdependent security games has shown that in most cases there is considerable difference to the global optimum. In most models, the level of investment at the equilibrium point of the game is less than optimal. For example, in the total effort interdependence model in [13], the equilibrium point is far below the global optimum of the game. Moreover, the presented model in [29] leads to a Nash equilibrium point where none of the players invests in their security. In [30] the existence of positive externalities is introduced as the reason of inefficiency of the equilibrium point. In other words, positive externalities are introduced in most of the researches as the main reason of the inefficiency of equilibrium points [31]. In the above mentioned articles, to eliminate adverse effects of externalities on players' security investments, it is advised to decrease the externalities. However, in the models that players possess limited security budgets and have to invest in their security, positive externalities will help them make optimal decisions and make better security investments by taking advantage of the effects of their neighbours' security investments [32]. There are also some other solutions like information sharing mechanisms, for certain conditions and under strict constraints. For instance, in [33] considering the behavior of strategic attackers, a mechanism based on sharing security information is presented to encourage players to invest at the social optimum level. However, in these methods, since sharing sensitive information of organizations can have negative effects on their privacy, the mechanism itself needs some incentives to be implemented which limits the effectiveness of these methods.

Although most of the literature has introduced positive externalities as the reason of inefficiency of equilibrium points, [34] showed that the externalities do not necessarily have adverse results on the optimality of equilibrium points. Therefore, it is required to investigate game design methods more carefully to encourage players to reach the social optimum point in their Nash equilibrium. In this paper we have presented an idea of optimizing the global security of the whole system using the game design method. Therefore, we have initially designed the game using the traditional design method of players' cost modifications. In the previous works the network has always been considered as priorly formed and fixed with very high-priced modifications [35, 36]. Whereas, in many cases like intra-organizational networks, modification of the network effects or players' constraints is possible. Therefore, in this paper we have presented a new approach to game design through influence graph and constraint modification.

4 Game Definition And Model

In order to model the problem we first formalize the underlying game $G(P, \mathbf{x}, \mathbf{U})$ in which $P = \{1, 2, \ldots, N\}$ is the set of N players, $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]$ is the vector of players' security investments and $\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_N]$ denotes the vector composed of their utility functions respectively. Players connect through a weighted directed network graph $Q = \{P, E\}$. Each node $i \in P$ corresponds to a selfish, non-malicious, intelligent and autonomous player. The edge set is represented as a matrix $E \in \{0, 1\}^{N \times N}$, which indicates interdependent links between the nodes in the network. The value 1 for $e_{ij} \in E$ demonstrates the existence of a link between nodes i and j and the value 0 implies the absence of such link in the graph. Each edge has got a weight $\mathbf{W}_{ij} \in \mathbb{R}$ representing the effect of the action of source node i on the destination node j (externalities) whose value is expressed as follows:

$$\begin{cases} \mathbf{W}_{ij} \in \mathbb{R} & \text{if } e_{ij} = 1 \text{ and } i \neq j \\ \mathbf{W}_{ij} = 1 & \text{if i = j} \\ \mathbf{W}_{ij} = 0 & \text{otherwise} \end{cases} \quad (2)$$

In the proposed model, each player i has a scalar valued investment $\mathbf{x}_i \in \mathbb{R}$. However, in most of the actual circumstances investments of players seem to be bounded. Furthermore, the effect of neighbours' security investments $\mathbf{x}_{-i}$ on node i as well as its own investment can be presented as $(\mathbf{W}^T x)_i$, which we denote for notational convenience as $(\mathbf{Mx})_i$ by redefining matrix $\mathbf{W}^T$ as $\mathbf{M}$. Then, the utility of each player i is defined as follows which is constrained regarding the security budget considerations imposed upon the players:

$$\begin{aligned} &\mathbf{U}_i(\mathbf{x}_i, \mathbf{x}_{-i}) = V_i\big((\mathbf{Mx})_i\big) - \mathbf{c}_i\mathbf{x}_i \\ &\text{subject to} \quad \mathbf{Ax} \leq \mathbf{b} \end{aligned} \quad (3)$$

In this formula matrix $\mathbf{A} \in \mathbb{R}^{r \times N}$ and vector $\mathbf{b} \in \mathbb{R}^r$ are used to determine the budget related constraints in which r illustrates the number of constraints. Therefore, for each player we have a concave utility maximization problem with a convex set of constraints.

In this formulation, $V_i(\cdot)$ is assumed to satisfy the following properties.

Assumption 4.1. $V_i(\cdot)$ is a twice continuous differentiable, strictly increasing $\frac{\partial}{\partial \mathbf{x}_i}V_i(\cdot) > 0$, strictly concave $\frac{\partial^2}{\partial \mathbf{x}_i^2}V_i(\cdot) < 0$ and twice differentiable function within the range $[0, \infty)$.

In the following for the notation convenience we denote $\frac{\partial V_i(\mathbf{x})}{\partial \mathbf{x}_i}$ as $V_i'(\cdot)$ and $\frac{\partial^2 V_i(\mathbf{x})}{\partial \mathbf{x}_i^2}$ as $V_i''(\cdot)$.

Table 1 represents summary of the notations along with a short description for each one.

Table 1. Summery of notations. More details can be found in appropriate sections.

Notation	Description
$G = \{P, \mathbf{x}, \mathbf{U}\}$	The game with the set $P = \{1, 2, \ldots, N\}$ of N players, vector $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]$ of players' security investments and vector $\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_N]$ of their utility functions.
$Q = \{P, E\}$	Weighted directed network graph with the set of $P = \{1, 2, \ldots, N\}$ nodes and edge set $E \in \{0, 1\}^{N\times N}$.
$\mathbf{W}$	Variable showing the weight matrix in which $\mathbf{W}_{ij} \in [0, 1]$ shows the effect of the source node i on its connection with the destination node j.
$\mathbf{M}$	An intermediate variable representing $\mathbf{W}^T$
$\mathbf{R_i}$	A variable representing resources used by node i
$\mathbf{A}$	Parameter $\mathbf{A} \in \mathbb{R}^{r\times N}$ is used to determine the budget related constraints.
$\mathbf{b}$	Parameter $\mathbf{b} \in \mathbb{R}^r$ is used to determine the budget related constraints in which r illustrates the number of constraints.
r	Parameter illustrates the number of constraints.
$\mathbf{x}$	Variable $\mathbf{x} \in \mathbb{R}^N_{\geq 0}$ shows a vector of investments of players in which $x_i \in \mathbb{R}_{\geq 0}$ shows the investment of player i
$\mathbf{c}$	A vector of players' costs. $\mathbf{c}_i$ shows cost of player i choosing investment $\mathbf{x}_i$.
$U_i(\mathbf{x_i}, \mathbf{x_{-i}})$	Total utility of player i based on its own investment and investments of all other players.
$V_i(.)$	The non-linear valuation function of player i based on its cumulative investments, containing its own investments and the effects of other players' investments.
μ	The Lagrangian of the social optimum problem.
λ	The Lagrangian of the Nash Equilibrium problem.

5 Game Design Using Cost Modification

In this section we consider a truthful complete information strategic security investment game in which players tend to optimize their utility functions as presented in (3). In such systems, the system owner usually looks for maximizing the total utility of different parts of the system regarding their security investments. However, generally there may exist differences between NE and the optimum point of the system from the system owner's point of view. So, the system owners look for methods to design system parameters and consequently encourage selfish players to reach a preferred point corresponding to a global social optimum. Therefore, in this section the goal is to approach the socially optimum point from a NE point using a market mechanism which modifies the cost functions of players through "pricing" signals. In other words, we propose an efficient mechanism in which the specific equilibrium point would be the same as the social optimum of the system.

Definition 5.1 (Efficiency). A mechanism or strategic game is said to be efficient if its outcome or the NE, $\mathbf{x}^*$, satisfies

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} \Big(\sum_{i=1}^{N} V_i\big((\mathbf{M}\mathbf{x})_i\big) \Big) \tag{4}$$

where the maximization of the sum of players valuations $\sum_{i=1}^{N} V_i\big((\mathbf{M}\mathbf{x})_i\big)$ regardless of their costs is the objective of the designer.

In the presented model in Section 4 there are three main parameters that can be modified for our game design purposes. Figure 2 depicts these and other relevant interdependent security parameters along with their relationships. NE points are affected by interdependencies which are shown as the network graph (matrix $\mathbf{M}$), players' prices c and their budgeting constraints (matrix $\mathbf{A}$). Furthermore, the social optimum also has a direct relationship with interdependencies as well as players' constraints. In this section, our focus is on approaching the social optimum using cost design or pricing signals, which is the classical approach prevalent in the literature. By extracting the indirect relationship between social optimum and NE points, a design method that brings the NE to the social optimum can be derived.

At first let us consider the **social welfare optimization problem** as a function of the aggregation of all user valuation functions:

$$\begin{aligned} \max_{\mathbf{x}} \quad & \sum_{i=1}^{N} V_i\big((\mathbf{M}\mathbf{x})_i\big) \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \tag{5}$$

We next write the Lagrangian for computing the social optimum,

$$\mathcal{L} = \sum_{i=1}^{N} V_i\big((\mathbf{M}\mathbf{x})_i\big) - \mu^T(\mathbf{A}\mathbf{x} - \mathbf{b}), \tag{6}$$

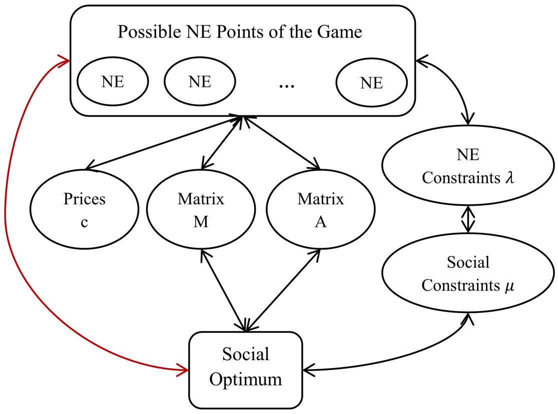

Figure 2. Effective parameters on the efficiency in an interdependent security game which can be used to move the NE to the social optimum.

and the following KKT conditions:

$$\begin{cases} (\mathbf{A}^T\mu)_i - \sum_{j=1}^{N} \mathbf{M}_{ji} V_j'((\mathbf{Mx})_j) = 0, & \forall i \in P \\ \mu_i^T((\mathbf{Ax})_i - \mathbf{b}_i) = 0, & \forall i \in P \\ \mathbf{Ax} \leq \mathbf{b} \\ \mu \geq 0 \end{cases} \tag{7}$$

Since $V_i : \mathbb{R} \to \mathbb{R}$ is concave and non-decreasing and $g(\mathbf{x}) = (\mathbf{Mx})_i$ as a function with **dom** $g = \mathbb{R}^N$ is also a concave function over its domain, then $V_i(g(\mathbf{x}))$ or $V_i((\mathbf{Mx})_i)$ is also a concave function. Therefore, the positive sum of these functions as the social welfare maximization (5) is a concave maximization problem over a convex set and hence the social optimum is a unique solution $(\hat{\mathbf{x}}, \hat{\mu})$.
Next, the relationship between the unique social optimum solution and the NE of the game is established. In other words, the social optimum and NE meet only when each player's best response corresponds to the social optimum point. Considering the players' utility functions as (3) the **best response** of each player $i \in P$ is as follows:

$$\begin{aligned} \arg\max_{\mathbf{x}_i} \quad & \mathbf{U}_i(\mathbf{x}) = V_i\big((\mathbf{Mx})_i\big) - \mathbf{c}_i\mathbf{x}_i \\ \text{subject to} \quad & \mathbf{Ax} \leq \mathbf{b} \end{aligned} \tag{8}$$

To compute the NE we should solve:

$$\max_{\mathbf{x}_i} \quad V_i\big((\mathbf{Mx})_i\big) - \mathbf{c}_i\mathbf{x}_i - \lambda^T(\mathbf{Ax} - \mathbf{b}) \tag{9}$$

leading to the following KKT conditions:

$$\begin{cases} V_i'((\mathbf{Mx})_i) - \mathbf{c}_i - (\mathbf{A}^T\lambda)_i = 0 \\ \lambda_i((\mathbf{Ax})_i - \mathbf{b}_i) = 0 \\ \mathbf{Ax} \leq \mathbf{b} \\ \lambda \geq 0 \end{cases} \tag{10}$$

It is worth mentioning that the game might have different NE. If there are multiple equilibrium points it is difficult and ambiguous to define an efficient mechanism. Therefore, to design an efficient mechanism some NE uniqueness sufficient conditions can be imposed. These conditions are presented in Appendix A. For the rest of the paper, in order to simplify the notation the Matrix σ is defined as:

$$\sigma = \begin{bmatrix} \sum_{j\neq 1}^{N} \mathbf{M}_{j1} V_j^{'}((\mathbf{M}\hat{\mathbf{x}})_j) \\ \sum_{j\neq 2}^{N} \mathbf{M}_{j2} V_j^{'}((\mathbf{M}\hat{\mathbf{x}})_j) \\ \vdots \\ \sum_{j\neq N}^{N} \mathbf{M}_{jN} V_j^{'}((\mathbf{M}\hat{\mathbf{x}})_j) \end{bmatrix} \tag{11}$$

Theorem 1 (Cost Design). Assume $\hat{\mathbf{x}}$ and $\hat{\mu}$ as the solution of social optimization problem presented in (5). If A is invertible and $\mathbf{M}$ and $\mathbf{V}$ satisfy the uniqueness conditions of NE, any modification of costs $\mathbf{c}$ to $\bar{\mathbf{c}}$ which belongs to the set defined by $S_c = \{\bar{\mathbf{c}} \in \mathbb{R}^N : \text{s.t. } (12) \text{ holds }\}$

$$\begin{cases} \bar{\lambda} = \hat{\mu} - \mathbf{A}^{T^{-1}}\sigma - \mathbf{A}^{T^{-1}}\bar{\mathbf{c}} \\ \bar{\lambda}((\mathbf{A}\hat{\mathbf{x}}) - \mathbf{b}) = 0 \\ \bar{\lambda} \geq 0 \end{cases} \tag{12}$$

leads to an efficient mechanism.

Proof. As depicted in Figure 2, costs of players' actions $\mathbf{c}$ is the only parameter which just affects the NE of the game (without changing the global optimum). Therefore, let us assume matrix $\mathbf{A}$ and $\mathbf{M}$ as fixed input parameters. Consequently, following (5) the social optimum can be computed as $\hat{\mathbf{x}}$ and $\hat{\mu}$ which is expected to be the same as the NE of players. Afterwards, by replacing $V_i'((\mathbf{M}\hat{\mathbf{x}})_i)$ in (10) with $(\mathbf{A}^T\hat{\mu})_i - \sum_{j\neq i}^{N} \mathbf{M}_{ji} V_j'((\mathbf{M}\hat{\mathbf{x}})_j)$ as a result of social optimization problem (7), $\bar{\mathbf{c}}$ can be computed as:

$$\bar{\mathbf{c}}_i = (\mathbf{A}^T\hat{\mu})_i - (\mathbf{A}^T\bar{\lambda})_i - \sum_{j\neq i}^{N} \mathbf{M}_{ji} V_j^{'}((\mathbf{M}\hat{\mathbf{x}})_j) \tag{13}$$

In this formula $\bar{\lambda}$ is an open variable and since changing $\bar{\mathbf{c}}$ could affect $\bar{\lambda}$, we are limited to choose $\bar{\mathbf{c}}$ in a way that makes $\bar{\lambda}$ non-negative. Therefore, by reformulating (13) and multiplying it by $\mathbf{A}^{T^{-1}}$, $\bar{\lambda}$ can be computed as follows:

$$\bar{\lambda}_i = \hat{\mu}_i - (\mathbf{A}^{T^{-1}}\bar{\mathbf{c}})_i - \mathbf{A}^{T^{-1}} \left(\sum_{j\neq i}^{N} \mathbf{M}_{ji} V_j^{'}((\mathbf{M}\hat{\mathbf{x}})_j) \right) \tag{14}$$

Finally the non-negativity check of $\bar{\lambda}$ leads us to (12) which is a set of linear equality (in non-boundary social optimum solutions where $\bar{\lambda} = 0$) and inequality (in boundary social optimum solutions where $\bar{\lambda} \geq 0$) of costs $\bar{\mathbf{c}}$. Therefore, we can formally define the set of costs ensuring an efficient mechanism as follows:

$$S_c = \{\bar{\mathbf{c}} \in \mathbb{R}^N : \text{s.t. } (12) \text{ holds }\} \tag{15}$$

□

Note that, any player's cost vector belonging to the set S_c can move the Nash Equilibrium point to the social optimum point.

Proposition 5.1 (Convexity Conditions). The set S_c in (15) is convex with at least a member.

Proof. Since (12) is a set of affine inequalities (in case $\mathbf{A}\hat{\mathbf{x}} = \mathbf{b}$) and equalities (in case $\mathbf{A}\hat{\mathbf{x}} \neq \mathbf{b}$) of $\bar{\mathbf{c}}$, the set S_c is a convex set. Furthermore, as $\bar{\mathbf{c}}$ is an unconstrained variable, in worst case which for all player i, $\bar{\lambda}_i = \hat{\mu}_i = 0$, (13) leads to a single result $\bar{\mathbf{c}}_i = -\sum_{j\neq i}^{N} \mathbf{M}_{ji} V_j'((\mathbf{M}\hat{\mathbf{x}})_j)$. □

In many cases the set S_c may have more than one members. Consequently, game design involves making a choice of costs from S_c. This selection process can be formulated as a convex optimization problem. As an example, we present the following optimization problem which aims to minimize the total costs of players:

$$\min_{\bar{\mathbf{c}}\in S_c} \quad \sum_i (\bar{\mathbf{c}}_i)^2 \tag{16}$$

5.1 Illustrative Numerical Example

As an illustrative numerical example, for the sake of simplicity and clarity and without loss of generalization, let us simply consider a 2 node fully connected network with

$$\mathbf{M} = \begin{bmatrix} 1 & 0.3 \\ 0.4 & 1 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}. \tag{17}$$

The valuation function is assumed to be $V(\cdot) = \log(\cdot)$. Therefore, the players utility function (3) is

$$\begin{aligned} \max_{\mathbf{x}_i} \quad & \log(\mathbf{M}_{ij}\mathbf{x}_j + \mathbf{x}_i) - \mathbf{c}_i\mathbf{x}_i \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \end{aligned} \tag{18}$$

By solving social optimum problem (5) we obtain $\hat{\mathbf{x}} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$ and $\hat{\mu} = \begin{bmatrix} 0.3467 \\ 0.0533 \end{bmatrix}$. Then, replacing $\hat{\mathbf{x}}$ and $\hat{\mu}$ in (12) the set S_c is as follows:

$$S_c = \{\bar{\mathbf{c}} \in \mathbb{R}^N : s.t. \begin{bmatrix} \bar{\mathbf{c}}_1 - \bar{\mathbf{c}}_2 \\ 2\bar{\mathbf{c}}_2 - \bar{\mathbf{c}}_1 \end{bmatrix} \leq \begin{bmatrix} 0.4667 \\ -0.2667 \end{bmatrix}\} \tag{19}$$

It is obvious that the set S_c is a convex set and applying the objective function (16) produces $\bar{\mathbf{c}}_1 = 0.0536$ and $\bar{\mathbf{c}}_2 = -0.1065$ as the result. Therefore, if we replace $\mathbf{c}_1$ and $\mathbf{c}_2$ in (18) for each player, their best response leads them to select $\hat{\mathbf{x}}_1 = 0$ and $\hat{\mathbf{x}}_2 = 5$ which concurrently maximizes their aggregate utilities or social welfare.

6 Game Design Using Interdependencies

In this section, we propose a method in which by modifying the externalities (software diversity in our example) players will be encouraged to reach the social optimum. In other words, we design the game by proposing a modification to matrix $\mathbf{M}$ while assuming that matrix $\mathbf{A}$ and costs $\mathbf{c}$ are fixed input parameters. However, since the weight matrix directly affects the social optimum, any modification of $\mathbf{M}$ might change the social optimum as well as the NE point. Therefore, we design the matrix $\mathbf{M}$ in such a way that ensures the original social optimum remains stable, whereas, the equilibrium of the game approaches to the original social optimum. We follow these three steps as part of the design process:

(1) Finding the social optimum of the game based on current structure of the graph.
(2) Modifying the externalities to ensure players' best responses will reach the social optimum.
(3) Making sure that this modification does not affect the social optimum itself.

Theorem 2 (Connection Design). Assume $\hat{\mathbf{x}}$ as the solution of optimization problem presented in (5) based on an initial weight matrix $\mathbf{M}$ and the corresponding game admits a unique NE. If V satisfies conditions in Assumption 4.1, any modification of matrix $\mathbf{M}$ to $\bar{\mathbf{M}}$ which satisfies:

$$\begin{cases} V_i'((\bar{\mathbf{M}}\hat{\mathbf{x}})_i) = (\mathbf{A}^T\bar{\lambda})_i + \mathbf{c}_i \\ \sum_{j=1}^{N} \bar{\mathbf{M}}_{ji} V_j'((\bar{\mathbf{M}}\hat{\mathbf{x}})_j) = (\mathbf{A}^T\bar{\mu})_i \\ \bar{\lambda}_i((A\hat{\mathbf{x}})_i - \mathbf{b}_i) = 0 \\ \bar{\mu}_i((\mathbf{A}\hat{\mathbf{x}})_i - \mathbf{b}_i) = 0 \\ \bar{\lambda}_i \geq 0 \\ \bar{\mu}_i \geq 0 \end{cases}, \forall i \in N \tag{20}$$

leads to an efficient game, i.e. NE coincides with the social optimum.

Proof. As mentioned above the convex social problem in (5) for each weight matrix $\mathbf{M}$ has a unique result. So, in the first step the game designer computes the social optimum $\hat{\mathbf{x}}$ based on current weight matrix $\mathbf{M}$. Then finding some $\bar{\mathbf{M}}$ which ensures that the given $\hat{\mathbf{x}}$ is also a unique NE of the game is needed. Therefore, $\hat{\mathbf{x}}$ is applied in KKT conditions in (10) which leads to the set $M_1 = \{\bar{\mathbf{M}} \in \mathbb{R}^{N\times N}$: s.t. (21) holds$\}$:

$$\begin{cases} V_i'((\bar{\mathbf{M}}\hat{\mathbf{x}})_i) = (\mathbf{A}^T\bar{\lambda})_i + \mathbf{c}_i \\ \bar{\lambda}_i((\mathbf{A}\hat{\mathbf{x}})_i - \mathbf{b}_i) = 0 \\ \bar{\lambda}_i \geq 0 \\ \mathbf{A}\hat{\mathbf{x}} \leq \mathbf{b} \end{cases}, \forall i \in N \tag{21}$$

Since $\hat{\mathbf{x}}$ is the result of social optimum based on $\mathbf{A}$ and $\mathbf{b}$, $\mathbf{A}\hat{\mathbf{x}} \leq \mathbf{b}$ always holds and can be eliminated. It is worth mentioning that although changing the weight matrix could affect the open variables $\bar{\lambda}$ and $\bar{\mu}$, the social objective $\hat{\mathbf{x}}$ must remain the same. Therefore, we need to guarantee that the new $\bar{\mathbf{M}}$ satisfies the conditions of social optimum problem and keep $\hat{\mathbf{x}}$; whereas, the value of Lagrangian coefficients of social optimum problem $\bar{\mu}$ might change. Hence, the pair of $\bar{\mathbf{M}}$ and $\hat{\mathbf{x}}$ should satisfy the KKT conditions of social optimum presented in (7) which leads to the set $M_2 = \{\bar{\mathbf{M}} \in \mathbb{R}^{N\times N} : \text{s.t. } (22) \text{ holds}\}$:

$$M_2 = \begin{cases} \sum_{j=1}^{N} \bar{\mathbf{M}}_{ji} V_j'((\bar{\mathbf{M}}\hat{\mathbf{x}})_j) = (\mathbf{A}^T\bar{\mu})_i \\ \bar{\mu}_i((\mathbf{A}\hat{\mathbf{x}})_i - \mathbf{b}_i) = 0 \\ \bar{\mu}_i \geq 0 \\ A\hat{\mathbf{x}} \leq \mathbf{b} \end{cases}, \forall i \in N \tag{22}$$

In this formula $\mathbf{A}\hat{\mathbf{x}} \leq \mathbf{b}$ can be eliminated by the same reason as above. Combining these two sets of constraints presented in (21) and (22) leads to S_M in (20). In other words, first group of constraints finds some matrix set M_1 which guarantees $\hat{\mathbf{x}}$ to be the NE of players and the second group of constraints finds some set M_2 ensuring the social optimum remains fixed. Consequently, if there exists an intersection between these sets, $S_M = \bar{\mathbf{M}} \subseteq M_1 \cap M_2$ the result/results will lead to an efficient equilibrium as:

$$S_M = \{\bar{\mathbf{M}} \in \mathbb{R}^{N\times N} : \text{s.t. } (20) \text{ holds}\} \tag{23}$$

□

6.1 Illustrative Numerical Example

In this section we consider an example network similar to the one presented in Section 5.1 with two nodes. We also assume similar initialization with

$$\mathbf{c}_1 = \mathbf{c}_2 = 0,\ \mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \tag{24}$$

The valuation function is also assumed as $V(\cdot) = \log(\cdot)$ with the same player's utility functions as in (18).

If we initially suppose $\mathbf{M}_{21} = 0.2$ and $\mathbf{M}_{12} = 0.1$ the social result will be $\hat{\mathbf{x}} = [6\ 4]^T$. By replacing $\hat{\mathbf{x}}$ in (20) and since $(\mathbf{A}\hat{\mathbf{x}})_i - \mathbf{b}_i = 0$ the constraints will change to:

$$\begin{cases} \bar{\lambda}_1 = \frac{2}{6+4\mathbf{M}_{12}} + \frac{1}{6\mathbf{M}_{21}+4} = 0 \\ \bar{\lambda}_2 = \frac{1}{6+4\mathbf{M}_{12}} + \frac{1}{6\mathbf{M}_{21}+4} \geq 0 \\ \bar{\mu}_1 = \frac{\mathbf{M}_{12}+2}{6+4\mathbf{M}_{12}} + \frac{2\mathbf{M}_{21}+1}{6\mathbf{M}_{21}+4} = 0 \\ \bar{\mu}_2 = \frac{\mathbf{M}_{12}+1}{6+4\mathbf{M}_{12}} + \frac{\mathbf{M}_{21}+1}{6\mathbf{M}_{21}+4} \geq 0 \\ \bar{\lambda}_1, \bar{\lambda}_2, \bar{\mu}_1, \bar{\mu}_2 \geq 0 \end{cases} \tag{25}$$

Therefore, we have $S_M = \{\bar{\mathbf{M}} \in \mathbb{R}^{N\times N}_{[0,1)} : \text{s.t. } (25) \text{ holds }\}$. It is evident that S_M is a convex set with infinitely many different solutions. However, if the constraints matrix changes to $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{c}_1$ to 10, then $\hat{\mathbf{x}} = \begin{bmatrix} 0.5698 \\ 0.4302 \end{bmatrix}$. Subsequently, checking the first line of (20) for $i = 1$ leads to $\frac{1}{\hat{\mathbf{x}}_1 + \mathbf{M}_{12}\hat{\mathbf{x}}_2} - 10 \geq 0$ which makes S_M to be an empty set and as a consequence, the problem will not admit any solutions. Hence, more accurate existence and convexity analysis of the set S_M is required.

6.2 Existence and Convexity Analysis

Proposition 6.1 (Convexity and Existence Conditions). If matrix $\mathbf{A}^T$ is invertible, $v(\bar{\mathbf{M}}\hat{\mathbf{x}}) \geq \mathbf{c} > 0$ and **a)** matrix $\mathbf{A}^{T^{-1}}$ and $\bar{\mathbf{M}}$ are positive or **b)** matrix $\mathbf{A}^{T^{-1}}$ and $\mathbf{A}^{T^{-1}}\bar{\mathbf{M}}^T$ are positive definite, then the set $S_{\mathbf{M}}$ in (23) is a convex set and if $r \leq N^2$ this set has at least a member.

Proof. Function $v(\mathbf{Mx})$ is defined as the pseudo-gradient of function V:

$$v(\mathbf{Mx}) = \begin{bmatrix} V_1'((\mathbf{Mx})_1) \\ V_2'((\mathbf{Mx})_2) \\ \vdots \\ V_N'((\mathbf{Mx})_N) \end{bmatrix} \tag{26}$$

To prove a). As stated earlier, both $\bar{\lambda}$ and $\bar{\mu}$ are free variables. So, we define the set of expected results S_M as the intersection of the set S_λ satisfying non-negativity of λ and a similar set S_μ satisfying non-negativity of μ. If $\mathbf{A}^T$ is invertible, then from the first row of (20) we obtain:

$$\bar{\lambda} = \mathbf{A}^{T^{-1}}\big(v(\bar{\mathbf{M}}\hat{\mathbf{x}}) - \mathbf{c}\big) \geq 0 \tag{27}$$

Since $\mathbf{A}^{T^{-1}} > 0$, both sides of the inequality can be multiplied by $\mathbf{A}^T$ to simplify the inequality to $v(\bar{\mathbf{M}}\hat{\mathbf{x}}) \geq \mathbf{c}$ such that:

$$S_\lambda = \{\bar{\mathbf{M}} \in \mathbb{R}^N : V_i'(\bar{\mathbf{M}}\hat{\mathbf{x}})_i \geq \mathbf{c}_i, \forall i \in N\} \tag{28}$$

To show that the set S_λ is convex, let assume $\mathbf{M}_1, \mathbf{M}_2 \in S_\lambda$. Therefore, $V_i'((\mathbf{M}_1\mathbf{x})_i) > \mathbf{c}_i$, $V_i'((\mathbf{M}_2\mathbf{x})_i) > \mathbf{c}_i$ and $(\mathbf{M}_1\hat{\mathbf{x}})_i \lessgtr (\mathbf{M}_2\hat{\mathbf{x}})_i$. So, for any $\alpha \in [0,1]$, $\mathbf{M}_3 = \alpha\mathbf{M}_1 + (1-\alpha)\mathbf{M}_2$ satisfies:

$$(\mathbf{M}_1\hat{\mathbf{x}})_i \lessgtr (\mathbf{M}_3\hat{\mathbf{x}})_i \lessgtr (\mathbf{M}_2\hat{\mathbf{x}})_i \tag{29}$$

Considering that V is a strictly increasing concave function based on Assumption 4.1, it is obvious that V' is a strictly decreasing function, so from (29) we have $V_i'((\mathbf{M}_3\mathbf{x})_i) > \mathbf{c}_i$ and the set S_λ is convex.

By the same token, the non-negativity of $\bar{\mu}$ should be investigated in the set S_μ. To this aim, using the invertability of $\mathbf{A}^T$ the second row of (20) will change to the following:

$$\bar{\mu} = \mathbf{A}^{T^{-1}} \bar{\mathbf{M}}^T v(\bar{\mathbf{M}}\hat{\mathbf{x}}) \geq 0 \tag{30}$$

We know that $v(\bar{\mathbf{M}}\hat{\mathbf{x}}) > 0$ and since $\mathbf{A}^{T^{-1}}$ and $\bar{\mathbf{M}}^T$ are positive, (30) always holds and $S_\mu = \{\bar{\mathbf{M}} \in R^N_{\geq 0}\}$. Consequently, if the intersection of two sets S_λ and S_μ is non-empty, the set S_M is a convex set with at least a member.

To prove b). If $v(\bar{\mathbf{M}}\hat{\mathbf{x}}) \geq \mathbf{c}$ and both sides of inequality in (27) are multiplied with $v(\bar{\mathbf{M}}\hat{\mathbf{x}}) - \mathbf{c}$, Since $\mathbf{A}^{T^{-1}}$ is positive definite, the inequality:

$$\left(v(\bar{\mathbf{M}}\hat{\mathbf{x}}) - \mathbf{c}\right)^T \mathbf{A}^{T^{-1}} \left(v(\bar{\mathbf{M}}\hat{\mathbf{x}}) - \mathbf{c}\right) \geq 0 \tag{31}$$

always holds. Therefore, S_λ is the same as (28) which as stated before is a convex set.

Moreover, since $v(\bar{\mathbf{M}}\hat{\mathbf{x}})$ is always non-negative in (30), both sides of inequality can be multiplied with it as:

$$v(\bar{\mathbf{M}}\hat{\mathbf{x}})^T \mathbf{A}^{T^{-1}} \bar{\mathbf{M}}^T v(\bar{\mathbf{M}}\hat{\mathbf{x}}) \geq 0 \tag{32}$$

So, because $\mathbf{A}^{T^{-1}} \bar{\mathbf{M}}^T$ is positive definite, the inequality always holds under the assumptions made. S_μ is convex since for any $\alpha \in [0, 1]$, we have

$$v(\bar{\mathbf{M}}\hat{\mathbf{x}})^T \left(\mathbf{A}^{T^{-1}} \alpha \bar{\mathbf{M}}^T + \mathbf{A}^{T^{-1}} (1-\alpha) \bar{\mathbf{M}}^T\right) v(\bar{\mathbf{M}}\hat{\mathbf{x}}) > 0 \tag{33}$$

Therefore, conditions **b** of Proposition 6.1 also support the convexity of the set S_M.

In both conditions **a** and **b** since $\mathbf{A}^{T^{-1}}$ is invertible and positive or positive definite, the non-negativity check of λ leads to $v(\bar{\mathbf{M}}\hat{\mathbf{x}}) \geq \mathbf{c}$. Because the number of variables N^2 is greater than or equal to the number of constraints, the set S_M is a non-empty set. □

When the set S_M has more than one members, game design involves choosing a specific weight matrix from this convex set. This selection process can be formulated as a constrained convex optimization problem. As an example, consider the objective of maximizing the effect of players' actions on the social objective over the convex set of constraints S_M. This is a concave maximization problem over a convex set which leads to a unique weight matrix:

$$\max_{\bar{\mathbf{M}} \in S_M} \sum_{i=1}^{N} V_i\big((\bar{\mathbf{M}}\hat{\mathbf{x}})_i\big) \tag{34}$$

If we have $\mathbf{M} = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$, social optimum leads to $\hat{\mathbf{x}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Solving (34) based on $\hat{\mathbf{x}}$, results in $\bar{\mathbf{M}} = \begin{bmatrix} 1 & 0.5 \\ 0.75 & 1 \end{bmatrix}$ which guaranties the equality of NE and social optimum with maximum possible amount.

7 Game Design Using Budgeting Constraints

In this section, we introduce an innovative method in which by modifying the budgeting constraints players will be encouraged to reach the social optimum. Firstly, we take the matrix $\mathbf{M}$ and costs $\mathbf{c}$ as fixed input parameters. The model works for weight matrices which are defined in a way that supports one of the sufficient conditions for uniqueness of NE presented in Appendix A. Therefore, as stated before the social optimum is also unique. Then, the game will be designed purely by modifying constraints $\mathbf{A}$ to new constraints $\bar{\mathbf{A}}$. Similar to designing interdependencies, this modification not only affects NE but it could also change social optimum as well. The objective is to design matrix $\bar{\mathbf{A}}$ to guarantee the equivalency of NE and the *original* social optimum by performing the following steps:

(1) Computing the social optimum regarding the current constraints.
(2) Modifying constraints to encourage the players' best responses to approach the specified point (social optimum point).
(3) Performing above actions as well as keeping the social optimum point stable.

Theorem 3 (Constraint Design). Assume $\hat{\mathbf{x}}$ as the solution of optimization problem presented in (5) based on initial constraints A and the corresponding game admits a unique NE. If V is invertible and also satisfies the conditions in Assumption 4.1, any modification of matrix $\mathbf{A}$ to $\bar{\mathbf{A}}$ which:

$$\begin{cases} (\bar{\mathbf{A}}^T\bar{\lambda})_i = V_i'((\mathbf{M}\hat{\mathbf{x}})_i) - \mathbf{c}_i \\ (\bar{\mathbf{A}}^T\bar{\mu})_i = \sum_{j=1}^{N} \mathbf{M}_{ji} V_j'((\mathbf{M}\hat{\mathbf{x}})_j) \\ \bar{\lambda}_i((\bar{\mathbf{A}}\hat{\mathbf{x}})_i - \mathbf{b}_i) = 0 \\ \bar{\mu}_i((\bar{\mathbf{A}}\hat{\mathbf{x}})_i - \mathbf{b}_i) = 0 & ,\forall i \in N \\ \bar{\lambda}_i \geq 0 \\ \bar{\mu}_i \geq 0 \\ \bar{\mathbf{A}}\hat{\mathbf{x}} \leq \mathbf{b} \end{cases} \tag{35}$$

leads to an efficient game, i.e. NE coincides with the social optimum.

Proof. First the game designer solves (5) to compute the unique $\hat{\mathbf{x}}$ based on initial constraints $\mathbf{A}$. Afterwards, these constraints are required to be modified to guarantee that NE will reach the specified point while social optimum remains the same. Therefore, $\hat{\mathbf{x}}$ is applied in KKT conditions of NE in (10) as follows:

$$\begin{cases} (\bar{\mathbf{A}}^T\bar{\lambda})_i = V_i'((\mathbf{M}\hat{\mathbf{x}})_i) - \mathbf{c}_i \\ \bar{\lambda}_i((\bar{\mathbf{A}}\hat{\mathbf{x}})_i - \mathbf{b}_i) = 0 \\ \bar{\lambda}_i \geq 0 \\ \bar{\mathbf{A}}\hat{\mathbf{x}} \leq \mathbf{b} \end{cases} , \forall i \in N \quad (36)$$

Furthermore, similar to the previous section, other constraints are needed to guarantee that the new $\bar{\mathbf{A}}$ satisfies the conditions of the social optimum problem and does not change it. In other words, by applying $\hat{\mathbf{x}}$ in KKT conditions of social optimum (7), we intend to keep the result stable while we adjust open variables $\bar{\lambda}$ and $\bar{\mu}$ as follows:

$$\begin{cases} (\bar{\mathbf{A}}^T\bar{\mu})_i = \sum_{j=1}^{N} \mathbf{M}_{ji}V_j'((\mathbf{M}\hat{\mathbf{x}})_j) \\ \bar{\mu}_i((\bar{\mathbf{A}}\hat{\mathbf{x}})_i - \mathbf{b}_i) = 0 \\ \bar{\mu}_i \geq 0 \\ \bar{\mathbf{A}}\hat{\mathbf{x}} \leq \mathbf{b} \end{cases} , \forall i \in N \quad (37)$$

If we define the sets $\mathbf{A}_1 = \{\bar{\mathbf{A}} \in \mathbb{R}^{r\times N} : \text{s.t. } (36) \text{ holds}\}$ and $\mathbf{A}_2 = \{\bar{\mathbf{A}} \in \mathbb{R}^{r\times N} : \text{s.t. } (37) \text{ holds}\}$, the set S_A is the result of $\mathbf{A}_1 \cap \mathbf{A}_2$ and can be represented as:

$$S_A = \{\bar{\mathbf{A}} \in \mathbb{R}^{r\times N} : \text{s.t. } (35) \text{ holds}\} \quad (38)$$

In this formula, r represents the number of constraints (rows of matrix $\mathbf{A}$) as defined in Section 4. □

7.1 Illustrative Numerical Example

Here to illustrate the influence of budgeting constraints on designing the game, we consider a similar network to the one presented in Section 5.1. The initial graph is chosen as a fully connected network with 2 nodes. We also let

$$\mathbf{c}_1 = \mathbf{c}_2 = 0.1, \mathbf{M} = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \quad (39)$$

The valuation function is also chosen as $V(\cdot) = \log(\cdot)$. Therefore, the players' utility function is the same as (18). If we initially assume $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ the social result will be $\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. By replacing $\hat{\mathbf{x}}$ in (35) the constraints change to:

$$\begin{cases} (\bar{\mathbf{A}}^T\bar{\lambda})_1 = \frac{1}{\hat{\mathbf{x}}_1 + 0.1\hat{\mathbf{x}}_2} - \mathbf{c}_1 = \frac{39}{110} \\ (\bar{\mathbf{A}}^T\bar{\lambda})_2 = \frac{1}{0.2\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2} - \mathbf{c}_2 = \frac{38}{120} \\ (\bar{\mathbf{A}}^T\bar{\mu})_1 = \frac{1}{\hat{\mathbf{x}}_1 + 0.1\hat{\mathbf{x}}_2} + \frac{0.2}{0.2\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2} = \frac{71}{132} \\ (\bar{\mathbf{A}}^T\bar{\mu})_2 = \frac{0.1}{\hat{\mathbf{x}}_1 + 0.1\hat{\mathbf{x}}_2} + \frac{1}{0.2\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2} = \frac{61}{132} \\ \bar{\lambda}_1, \bar{\lambda}_2, \bar{\mu}_1, \bar{\mu}_2 \geq 0 \\ \bar{\lambda}(\bar{\mathbf{A}}\hat{\mathbf{x}} - \mathbf{b}) = 0 \text{ and } \bar{\mu}(\bar{\mathbf{A}}\hat{\mathbf{x}} - \mathbf{b}) = 0 \end{cases} \quad (40)$$

Note that, we obtain in this special case a convex set with infinitely many different matrices $\bar{\mathbf{A}}$ satisfying the above constraints. However, this may not be the case for other parameter choices. Therefore, the set S_A is further analyzed next.

7.2 Existence and Convexity Analysis

Proposition 7.1 (Convexity and Existence Conditions). The set S_A specified in (38) is a convex set with at least one solution, if $\bar{\mathbf{A}}^T$ is invertible and $N \geq 2$.

Proof. To prove this, we just need to multiply the first two constraints in (35) with $\bar{\mathbf{A}}^{T-1}$. Therefore, if we consider $v(\mathbf{M}\mathbf{x})$ as defined in (26), then (35) changes to:

$$\begin{cases} \bar{\lambda} = \bar{\mathbf{A}}^{T-1}(v(\mathbf{M}\hat{\mathbf{x}}) - \mathbf{c}) \\ \bar{\mu} = \bar{\mathbf{A}}^{T-1}(\mathbf{M}^T v(\mathbf{M}\hat{\mathbf{x}})) \\ \bar{\lambda}((\bar{\mathbf{A}}\hat{\mathbf{x}}) - \mathbf{b}) = 0 \\ \bar{\mu}((\bar{\mathbf{A}}\hat{\mathbf{x}}) - \mathbf{b}) = 0 \\ \bar{\lambda} \geq 0 \\ \bar{\mu} \geq 0 \end{cases} \quad (41)$$

As a matter of fact, (41) represents two inequalities $\bar{\mathbf{A}}^{T-1}(v(\mathbf{M}\hat{\mathbf{x}}) - \bar{\mathbf{c}}) \geq 0$ and $\bar{\mathbf{A}}^{T-1}(\mathbf{M}^T v(\mathbf{M}\hat{\mathbf{x}})) \geq 0$ except for the boundary points in which these inequalities change to equalities. Consequently, since the set S_A contains only some linear equalities and inequalities in $\bar{\mathbf{A}}^{T-1}$, it is a convex set.

For the non-negativity check of λ and μ we have $2r$ equality(non-boundary points) or inequality(boundary points) constraints for both of them and $r \times N$ variables (elements of matrix $\bar{\mathbf{A}}$). Therefore, if $N \geq 2$, then the set S_A has at least a member. □

Since different matrices $\bar{\mathbf{A}}$ may satisfy above conditions, another objective function can be added to pick out one of them. One interesting objective function could be selecting the least level of constraints:

$$\begin{aligned} \min_{\bar{\mathbf{A}} \in S_A} \quad & \sum_{i,j} (\bar{\mathbf{A}}_{ij})^2 \\ \text{subject to} \quad & \bar{\mathbf{A}}^T \text{ is invertible and } N \geq 2 \end{aligned} \quad (42)$$

It is clear that the objective function is a convex function of $\bar{\mathbf{A}}$ under the set of convex constraints S_A, and hence the problem admits a unique solution.

8 Case Study

In this section we reconsider the case study presented in Section 2 to show the applicability of the proposed methods numerically. For this purpose, we model the mentioned case as a game and apply three different proposed design methods to move Nash Equilibrium of the game to social optimum. Then we compare the gains achieved by different mechanisms regarding their corresponding ease, cost, and feasibility for the system owner to facilitate his/her decision making process.

As it is presented in Section 2, a company with four different sections is considered which uses its own private service provider. Figure 1 presents interdependencies of four different sections of the mentioned organization whose interdependencies are formed based on formula (1) and regarding their usage of shared memory presented by their private service provider. In the presented example, since players are sections of a specific organization, unit costs of players' investments are the same. Furthermore, the budget related constraints are presented as the following rules:

- The total amount of security investments of sections 1, 2 and 3 can at most be equal to \$7000.
- The total amount of security investments of sections 2 and 4 can at most be equal to \$3000.
- Section 3 can invest at most \$1000 more than section 2.
- Section 4 can invest at most \$2000 more than section 2.

Considering the game definition presented in section 3, the above security investment constraints can be represented as $Ax <= b$. Therefore, unit costs of players' investments, budget related constraints, and weight matrix can be represented as follows. All monetary units are considered in thousand dollars (i. e. c_i, b_i and x_i).

$$\mathbf{W} = \begin{bmatrix} 1 & 0.2 & 0.1 & 0.2 \\ 0.1 & 1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 1 \end{bmatrix} \ \& \ \mathbf{c} = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$$
$$\mathbf{A} = \begin{bmatrix} 1 & -0.2 & 0.5 & -0.3 \\ 0.2 & 1 & 0.3 & -0.5 \\ -0.2 & 0.1 & 1 & 0.1 \\ 0.3 & -0.2 & -0.1 & 1 \end{bmatrix} \ \& \ \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \quad (43)$$

Since players are parts of a large company, we also consider function $V(.) = log(.)$ for all players. Considering the definition of utility function in formula (3), selfish behavior of players leads to the Nash Equilibrium as follows:

$$\hat{\mathbf{x}}_{Nash} = \begin{bmatrix} 2 \\ 2.2777 \\ 1 \\ 1.9555 \end{bmatrix} \quad (44)$$

Moreover, solving the social optimum problem represented in formula (5) leads to the following security investments of players:

$$\hat{\mathbf{x}}_{SO} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \ \& \ \hat{\mu}_{SO} = \begin{bmatrix} 0.2095 \\ 0.6521 \\ 0.3054 \\ 0.8326 \end{bmatrix} \quad (45)$$

After solving the Nash Equilibrium and social optimum of the game, considering the price of anarchy $PoA = 0.8777$, reveals the difference between the total utility of players in equilibrium point and their total utility in global optimum. Therefore, to move the equilibrium point to the social optimum, the presented game design mechanisms are applied. Making use of the first game design method, presented in this chapter, leads to a set of costs for players to reach the global optimum point. It is worth noting that using formula (18) to select the lowest cost from the cost set in the game design, the matrix $\bar{\mathbf{c}}$ can be calculated as follows:

$$\bar{\mathbf{c}} = \begin{bmatrix} 0.3802 \\ 0.1150 \\ 0.0248 \\ 0.3970 \end{bmatrix} \quad (46)$$

In other words, if the network owner apply the calculated costs $\bar{\mathbf{c}}$, the best security investment of the players in their equilibrium point $\hat{\mathbf{x}}_{Nash}$ would be equivalent to the matrix presented in formula (45). The total costs imposed upon the players which can be considered as the game designer's income, is equal to 1.1552. Furthermore, based on the second method of game design presented in Section 6, the change of weight matrix in the given example can also lead players to reach the global optimum point selecting their selfish choices. Using (34) for selecting among possible weights in this game design method, the matrix $\bar{\mathbf{W}}$ is calculated as:

$$\bar{\mathbf{W}} = \begin{bmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix} \quad (47)$$

In other words, applying interdependency modification mechanism changes the interdependency graph to one presented in Figure 3. In the presented case, when the company's service provider for some sections like sections 3 and 4, as well as sections 2 and 4 chooses to completely share the memory without any recovery considerations, and for other pairs of players, considers recovery of some portion of their memory, it will result in the social optimum of the game. For example, in the computed weight matrix, while completely shared memory is considered for both players ($\frac{|\mathbf{R}_i \cap \mathbf{R}_j|}{|\mathbf{R}_i \cup \mathbf{R}_j|} = 1$), the mechanism designer guarantees to recover %25 of the memory of player j ($RecoveryGuarantee(\mathbf{R}_j) = 0.25$). Therefore, applying formula (1) leads to the interdependencies of Figure 3.

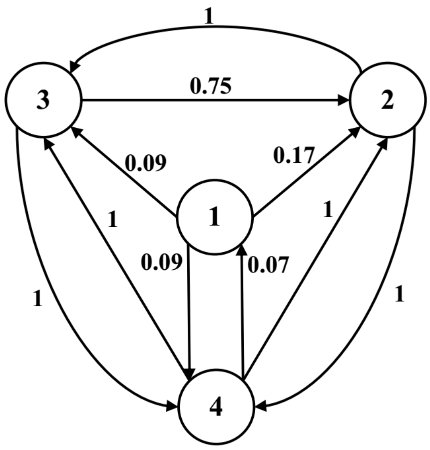

Figure 3. The result of interdependency modification mechanism.

If we consider the unit cost of increasing or decreasing the underlying infrastructure resources (memory) as $WCost$ and the cost of applying recovery mechanisms for each unit of infrastructure resources as $RCost$, the benefit of the interdependency change in the network graph from the designer's point of view can be calculated using the following formula:

$$\begin{aligned} ChangeBenefit = &\sum_{i,j \in N, i \neq j} (Max(\bar{\mathbf{W}}_{ij}, \bar{\mathbf{W}}_{ji}) \\ &- Max(\mathbf{W}_{ij}, \mathbf{W}_{ji})) \times WCost + \\ &(|\ \mathbf{W}_{ij} - \mathbf{W}_{ji}\ | - |\ \mathbf{W}_{ij} - \mathbf{W}_{ji}\ |) \times RCost \end{aligned} \quad (48)$$

In the above formula, the first term represents the benefits resulting from the increase/decrease of the shared resources used in each pair of nodes i and j, and the second term presents the benefits resulting from the change in the application of recovery considerations. In the present case, the unit cost of the application of recovery mechanisms for each unit of infrastructure resources is considered twice as high as the unit cost of changing shared infrastructure resources as $RCost = 2 \times WCost$. Figure 4 shows the benefits derived from the application of cost modification and interdependency modification mechanisms on the applied case if the unit cost $WCost$ increases from 0.01 to 0.1, from the designer's point of view. As indicated in Figure 4, if $WCost < 0.013$, the cost modification mechanism will cause greater benefit for the game designer and for $Wcost > 0.013$ interdependency modification mechanism will cause more benefits than the cost modification mechanism.

Finally, in order to achieve the global optimum at the equilibrium point, by employing the objective function (42), change of matrix A as follows, motivates the selfish players to achieve the global optimum point at their equilibrium point.

$$\bar{\mathbf{A}} = \begin{bmatrix} 0.4307 & 0.5190 & -0.8775 & 0.9279 \\ 0.2295 & 0.3075 & 0.3915 & 0.0714 \\ 0.2253 & 0.1333 & 0.5183 & 0.1231 \\ 0.2961 & 0.1104 & 0.2008 & 0.3927 \end{bmatrix} \quad (49)$$

The owner of the network will now be able to choose among possible mechanisms, with respect to ease, cost and feasibility.

9 Conclusion

In the presented paper three different game design methods are proposed using parameter adjustments in interdependent networks and are clarified using a real-world case of an organization's network as well as some illustrative examples. The case shows a network of nodes with some levels of shared resources which makes their interdependencies. In this networks players tend to maximize their utilities regarding their interdependencies and their limited security budgets. Therefore, they invest their money to apply some security mechanisms. However, these mechanisms do not guarantee the social objective to be optimum. Therefore, the network owner makes players reach the optimum point by adding some costs to them, modifying their interdependencies or changing the players' budgeting constraints. In other words, in addition to the classical approach of modifying players' costs through price signals to reach the social optimum, the same objective is achieved by direct modification of interdependencies as well as the budgeting con-

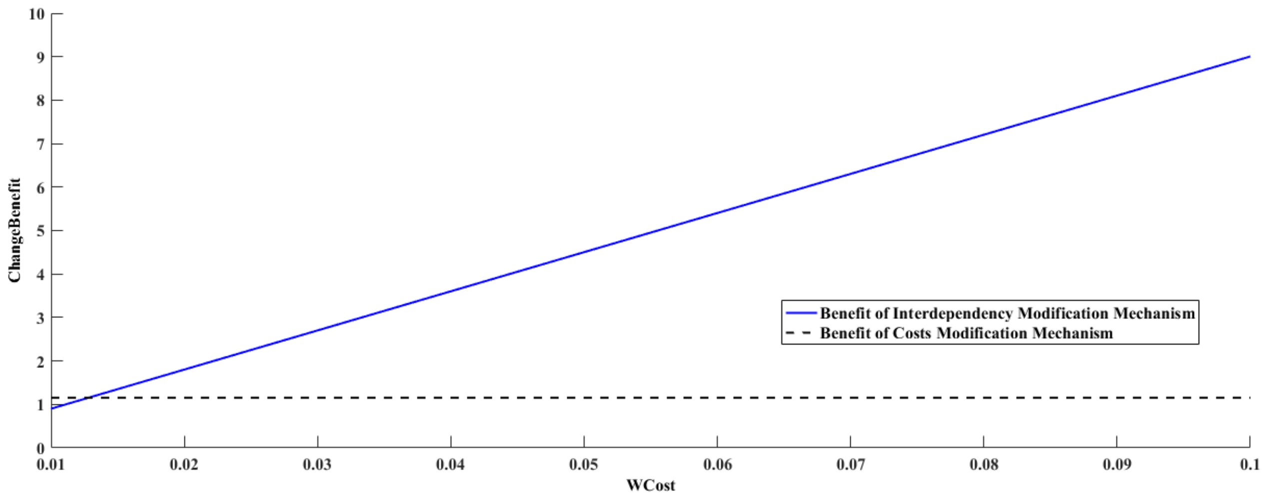

Figure 4. Comparing the Benefits of cost modification Mechanism and interdependency modification mechanism.

straint set. Furthermore, some sufficient conditions for existence and uniqueness of NE considering general constraints are also presented.

This framework provides network owners some interesting and useful alternatives for achieving the maximum possible social welfare. The presented approach can be, for example, more useful for intra-organizational networks when the modification of interdependencies or constraints might be more efficient in comparison to imposing some costs upon players.

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10 Appendix

A NE Analysis of Games on Linear Influence Networks under General Constraints

As discussed in Section 3, several recent studies have investigated Nash Equilibrium solution and its properties in unconstrained games played on networks. In these studies the utility function of each player also corresponds to a concave function presented in (3) without or with very simple constraints and the existence and uniqueness of NE has been established under various sufficient conditions. On the other hand, in this paper we have considered the problem under the general constraints as presented in (3) on linear influence networks.

In the following we investigate the existence and uniqueness of the equilibrium point for the presented strategic game $G(N, S, U)$, where N, S, and U are set of players, players strategies, and their utility functions respectively. First, we present the following well-known existence theorem for completeness.

Theorem 4 (Existence of NE). All constrained strategic network games on linear influence networks with utility function presented in (3) admit a NE solution.

Proof. As mentioned in Section 4 the strategy of each player i is a scalar $x_i \in \mathbb{R}$ which is bounded by the constraint $Ax \leq b$. Therefore, the players' strategy sets are both convex and compact. Furthermore, the utility function $U_i(x_i, x_{-i})$ is also a continuous function of x_{-i} and is continuous and concave (hence also quasi-concave) based on Assumption 4.1. Since players' strategies are infinite and

(1) The strategy set of each player is a convex and compact set.
(2) The utility function is a continuous function for the actions of other players.
(3) The utility function is a continuous and quasi-concave function for its own actions.

as it is proved in [37–39] the game has a pure NE: □

We next present some sufficiency theorems for uniqueness of the Nash Equilibrium points of the game.

Proposition A.1 (Uniqueness of NE). In constrained strategic network games on linear influence networks with the utility function presented in (3), if we assume $x \in \mathbb{R}^N_{\geq 0}$ and for every pair of strategy profiles $x^a, x^b \in S$, $(x^a_i - x^b_i) \times \big(M(x^a - x^b)\big)_i > 0, \forall i \in N$ holds, then the game has a unique NE point.

Proof. If for every pair of strategy profiles $x^a, x^b \in S$, $(x^a_i - x^b_i) \times \big(M(x^a - x^b)\big)_i > 0, \forall i \in N$, regarding the characteristics of concave function $V(\cdot)$ we have:

$$(x^a_i - x^b_i) \times \Big(V'_i\big((Mx^b)_i\big) - V'_i\big((Mx^a)_i\big)\Big) > 0 \quad \text{(A.1)}$$

And therefore:

$$\begin{bmatrix} x^a_1 - x^b_1 & x^a_2 - x^b_2 & \dots & x^a_N - x^b_N \end{bmatrix} \times \begin{bmatrix} V'_1((Mx^b)_1) - C_1 - V'_1((Mx^a)_1) + C_1 \\ V'_2((Mx^b)_2) - C_2 - V'_2((Mx^a)_2) + C_2 \\ \vdots \\ V'_N((Mx^b)_N) - C_N - V'_N((Mx^a)_N) + C_N \end{bmatrix} > 0 \quad \text{(A.2)}$$

If $\nabla u(x)$ is defined as $[\nabla_1 u_1(x), \dots, \nabla_N u_N(x)]^T$, since it is proved that:

$$(x^a - x^b)^T \big(\nabla u(x^b) - \nabla u(x^a)\big) > 0 \quad \text{(A.3)}$$

The utility functions $(u_1, \dots, u_{|N|})$ are diagonally strictly concave on $x \in S$. Therefore, considering the assumption $x \geq 0$, as it is proved by Rosen in [40] the game has a unique NE solution. □

Theorem 5 (Uniqueness of NE). In constrained strategic network games on linear influence networks with utility function presented in (3), if for each strategy profile $x \in S$, $|V_i^{''}((Mx)_i)| > \frac{1}{2}\sum_{j\neq i} |M_{ij}^T V_i^{''}((Mx)_i) + M_{ji}^T V_j^{''}((Mx)_j)|, \forall i$, then the game has a unique NE solution.

Proof. We first redefine the concave utility maximization problem defined in (3) to the following convex cost minimization problem.

$$\begin{aligned} \min_{x_i} \quad & J_i(x) = c_i x_i - V_i\big((Mx)_i\big) \\ \text{subject to} \quad & Ax \leq b \end{aligned} \tag{A.4}$$

Hence, we have $d(x)$ as follows:

$$\begin{aligned} d(x) = & \left[\frac{\partial J_1(x)}{\partial x_1} \ \frac{\partial J_2(x)}{\partial x_2} \ \dots \ \frac{\partial J_N(x)}{\partial x_N}\right] = \\ & \left[c_1 - V_1'((Mx)_1) \ \ c_2 - V_2'((Mx)_2) \ \dots \ c_N - V_N'((Mx)_N)\right] \end{aligned} \tag{A.5}$$

And consequently $D(x)$ is presented as:

$$D(x) = \begin{bmatrix} \frac{\partial^2 J_1(x)}{\partial x_1^2} & \frac{\partial^2 J_1(x)}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 J_1(x)}{\partial x_1 \partial x_N} \\ \frac{\partial^2 J_2(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 J_2(x)}{\partial x_2^2} & \dots & \frac{\partial^2 J_2(x)}{\partial x_2 \partial x_N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2 J_N(x)}{\partial x_N \partial x_1} & \frac{\partial^2 J_N(x)}{\partial x_N \partial x_2} & \dots & \frac{\partial^2 J_N(x)}{\partial x_N^2} \end{bmatrix} = -1 \times$$

$$\begin{bmatrix} V_1^{''}((Mx)_1) & M_{12}V_1^{''}((Mx)_1) & \dots & M_{1N}V_1^{''}((Mx)_1) \\ M_{21}V_2^{''}((Mx)_2) & V_2^{''}((Mx)_2) & \dots & M_{2N}V_2^{''}((Mx)_2) \\ \vdots & \vdots & \vdots & \vdots \\ M_{N1}V_N^{''}((Mx)_N) & M_{N2}V_N^{''}((Mx)_N) & \dots & V_N^{''}((Mx)_N) \end{bmatrix} \tag{A.6}$$

Therefore, since $V_i^{''}((Mx)_i) < 0$ and $|V_i^{''}((Mx)_i)| > \frac{1}{2}\sum_{j\neq i} |M_{ij}^T V_i^{''}((Mx)_i) + M_{ji}^T V_j^{''}((Mx)_j)|, \forall i$, the Hermitian matrix $F(x) = D(x) + D(x)^T$ is strictly diagonally dominant and as a consequence is positive definite. Therefore, applying the NE uniqueness theorem presented in [41] the game has a unique NE. □